
Student Number

North Sydney Girls High School

2024 **HSC TRIAL EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8–15)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

NAME: ______________________ **TEACHER:** ____________________

Question	1-10	11	12	13	14	Total
Total	/10	/ 15	/15	/15	/15	/70

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following gives the solutions to the inequality $|2 - 3x| \leq 4$?

A. $-\frac{2}{3} \leq x \leq 2$

B. $-2 \leq x \leq \frac{2}{3}$

C. $x \leq -2, \quad x \geq \frac{2}{3}$

D. $x \leq -\frac{2}{3}, \quad x \geq 2$

2 A curve is defined parametrically by the equations $x = 2\cos t$ and $y = 2\cos 2t$.
What is the Cartesian of the curve?

A. $y = 2 + x^2$

B. $y = x^2 - 2$

C. $y = 2x$

D. $y = 2x^2 - 1$

3 Which of the following is the correct expression for $\int \sin^2 3x\, dx$?

A. $\frac{1}{2}(1 - \cos 6x) + C$

B. $x - \frac{1}{6}\sin 6x + C$

C. $\frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right) + C$

D. $\frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + C$

4 Which of the following is the correct value for $\cos^{-1}(-p) + \sin^{-1}(-p)$ where $-1 \leq p \leq 1$?

A. $-\frac{\pi}{2}$

B. 0

C. $\frac{\pi}{2}$

D. π

5 Given that $f(x) = x^2 - 4x + 2$, where $x \geq 2$, which of the following is $f^{-1}(x)$?

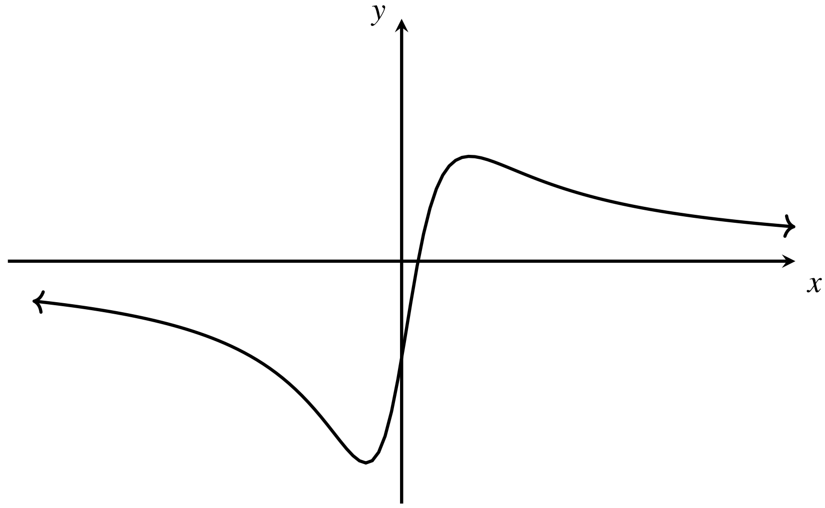
A. $f^{-1}(x) = 2 \pm \sqrt{x+2}$

B. $f^{-1}(x) = 2 + \sqrt{x+2}$

C. $f^{-1}(x) = 2 - \sqrt{x+2}$

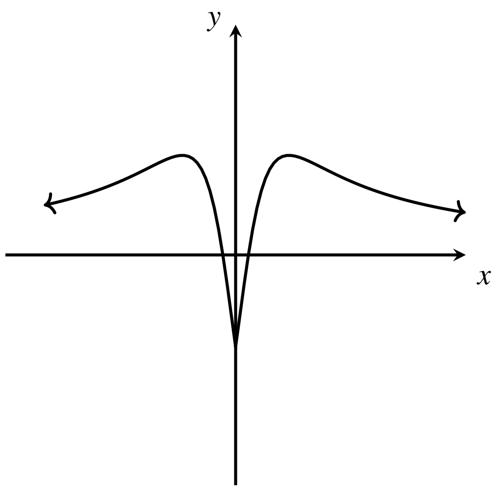
D. $f^{-1}(x) = 2 - \sqrt{x-2}$

6 A sketch of the graph of $y = f(x)$ is given below.

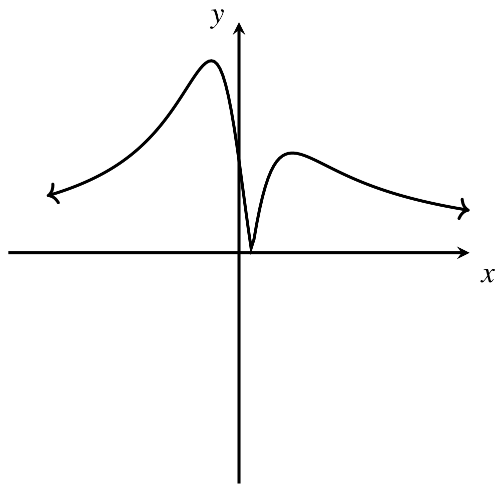


Which of the following is the graph of $y = \dfrac{1}{f(x)}$?

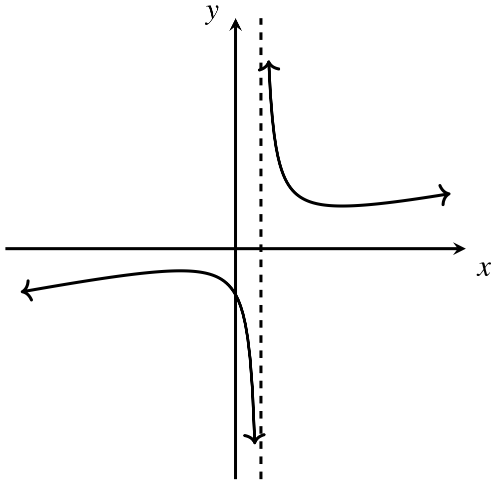
A.



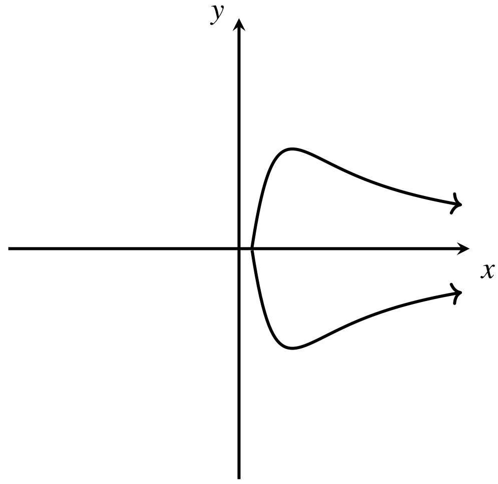
B.



C.



D.



7 If points A, B and C are such that $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, which of the following statements must be true?

A. Either $\overrightarrow{AB}$ or $\overrightarrow{BC}$ is the zero vector.

B. Points A, B and C are collinear.

C. The vector projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$ is $\overrightarrow{AB}$.

D. The vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$ is $\overrightarrow{AC}$.

8 A set of five distinct parallel lines intersects a different set of seven distinct parallel lines. How many possible parallelograms can be made from these lines?

A. ${}^{12}C_4$

B. ${}^{5}C_2 \times {}^{7}C_2$

C. ${}^{5}C_2 + {}^{7}C_2$

D. ${}^{5}P_2 \times {}^{7}P_2$

9 The graph of $y = \sqrt{f(|x|)}$ is not defined for any value of x. In which quadrants can the graph of $y = f(x)$ exist?

A. Fourth only

B. First and Second

C. Third and Fourth

D. Second, Third and Fourth

10 The letters of the word IMPRESSION are rearranged such that the letters of the word PRESS appear in order but not necessarily together.

One such arrangement is O **P** N **R** **E** I **S** M **S** I.

How many such arrangements are there?

A. $\dfrac{10!}{2!2!}$

B. $\dfrac{10!}{5!2!}$

C. 6!

D. 10!

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE Writing Booklet

(a) Solve $\dfrac{x+1}{x-3} \leq 2$. **3**

(b) It is given that $\mathbf{u} = 5\mathbf{i} - 4\mathbf{j}$ and $\mathbf{v} = 8\mathbf{i} + 9\mathbf{j}$.

(i) Find $\mathbf{u} \cdot \mathbf{v}$. **1**

(ii) Hence, explain with reasons whether the angle between $\mathbf{u}$ and $\mathbf{v}$ is acute or obtuse. **1**

(c) (i) Show that $2\cos 2x \cos 3x = \cos x + \cos 5x$. **1**

(ii) Hence, solve the equation $\cos x + \cos 5x = 0$ for $0 \leq x \leq \pi$. **2**

(d) A polynomial is given such that $p(x) = x^3 - 8x^2 + qx - 16$ where $q \in \mathbb{R}$.
It is known that the equation $p(x) = 0$ has roots α, β and $\alpha + \beta$.

(i) Show that $\alpha = 2$ and $\beta = 2$. **2**

(ii) Hence find the value of q. **1**

Question 11 continues on page 9

(e) A coastguard stationed at the origin O monitors the movements of a small boat which is moving in a straight line at a constant speed measured in km/h. The coastguard spots the boat at $(7\mathbf{i} - \mathbf{j})$ at 10 a.m. At noon the location of the boat is at $(-\mathbf{i} - 5\mathbf{j})$.
Each unit is 1 km.

(i) Find the velocity vector of the boat. **1**

(ii) Hence, if t is the time in hours after 10 a.m., find the displacement vector for the position of the boat at time t. **1**

(iii) Hence, find the value of t when the boat is closest to the coastguard. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE Writing Booklet

(a) Sketch the graph of the function $f(x) = \cos^{-1}(2x - 1)$. **2**

(b) Find $\displaystyle\int \frac{1}{\sqrt{9 - 4x^2}}\,dx$. **2**

(c) Prove the following identity: **3**

$$\frac{\tan a}{\tan 2a - \tan a} = \cos 2a$$

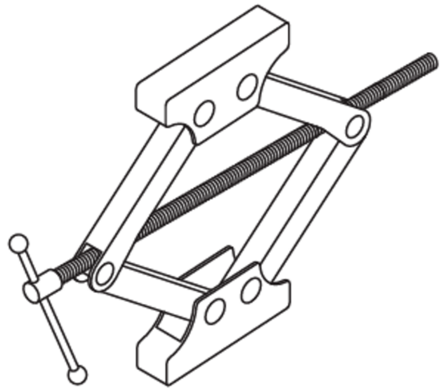
(d) (i) Express $\sqrt{3}\sin 3t - \cos 3t$ in the form $R\sin(3t - \alpha)$ where α is acute and $R > 0$. **2**

(ii) Hence, or otherwise, solve $\sqrt{3}\sin 3t - \cos 3t = 0$ for $0 \le t \le \pi$. **2**

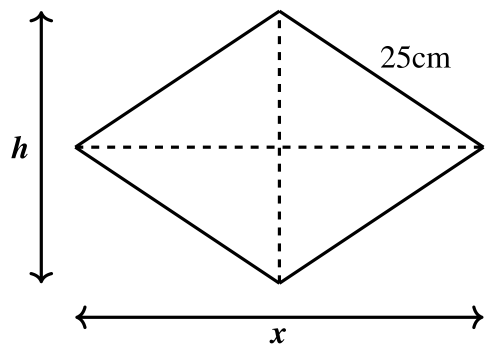
Question 12 continues on page 11

Question 12 (continued)

(e) An engineer has designed a lifting device. The handle turns a screw which shortens the horizontal length and increases the vertical height.



The device is modelled by a rhombus of side length 25 cm.
The horizontal diagonal of the rhombus is x cm, and the vertical diagonal is h cm as shown below.



(i) Show that $h = \sqrt{2500 - x^2}$. **2**

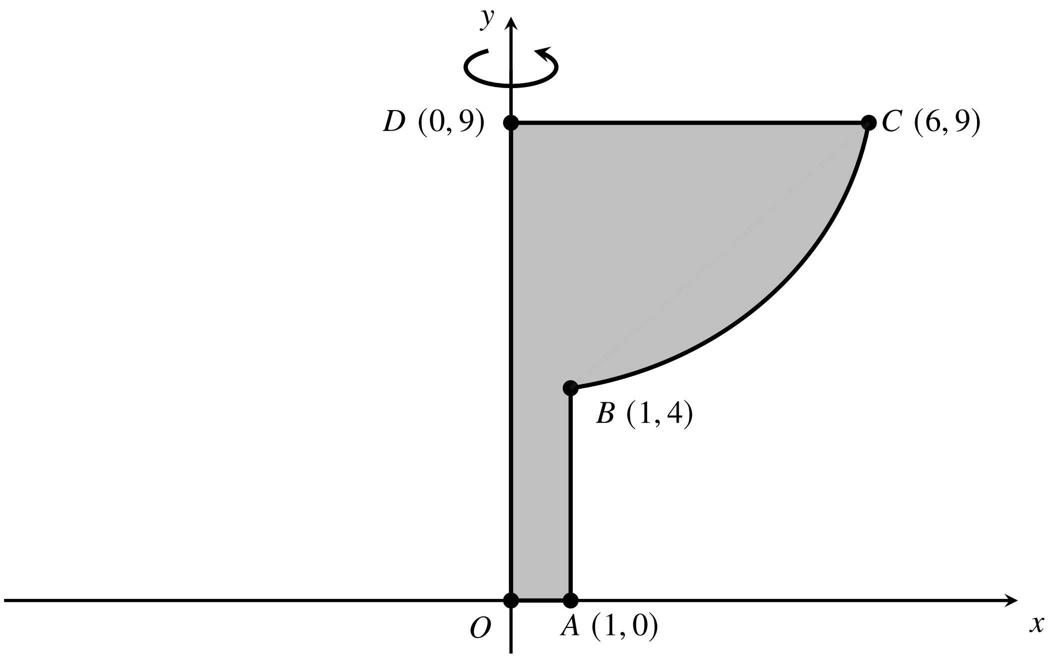
(ii) As the handle is turned, the length of the horizontal diagonal decreases at a constant rate of 0.3 cm per second. **2**
Find the rate of change of the vertical diagonal when the horizontal diagonal is 30 cm long.

End of Question 12

Question 13 (15 marks) Use a SEPARATE Writing Booklet

(a) O is the origin of a coordinate system whose units are in cm. The points A, B, C and D have coordinates $(1, 0)$, $(1, 4)$, $(6, 9)$ and $(0, 9)$ respectively. The arc BC is part of the curve with equation $x^2 + (y - 10)^2 = 37$. The closed shape $OABCD$ is formed, in turn, from the line segments OA and AB, the arc BC and the line segments CD and DO (see diagram). **3**

Find the exact volume of the object formed when $OABCD$ is rotated about the y-axis.



(b) Using the substitution $u = \ln x$, evaluate $\displaystyle\int_1^e \frac{\sqrt{1 + \ln x}}{2x}\,dx$. **3**

Question 13 continues on page 13

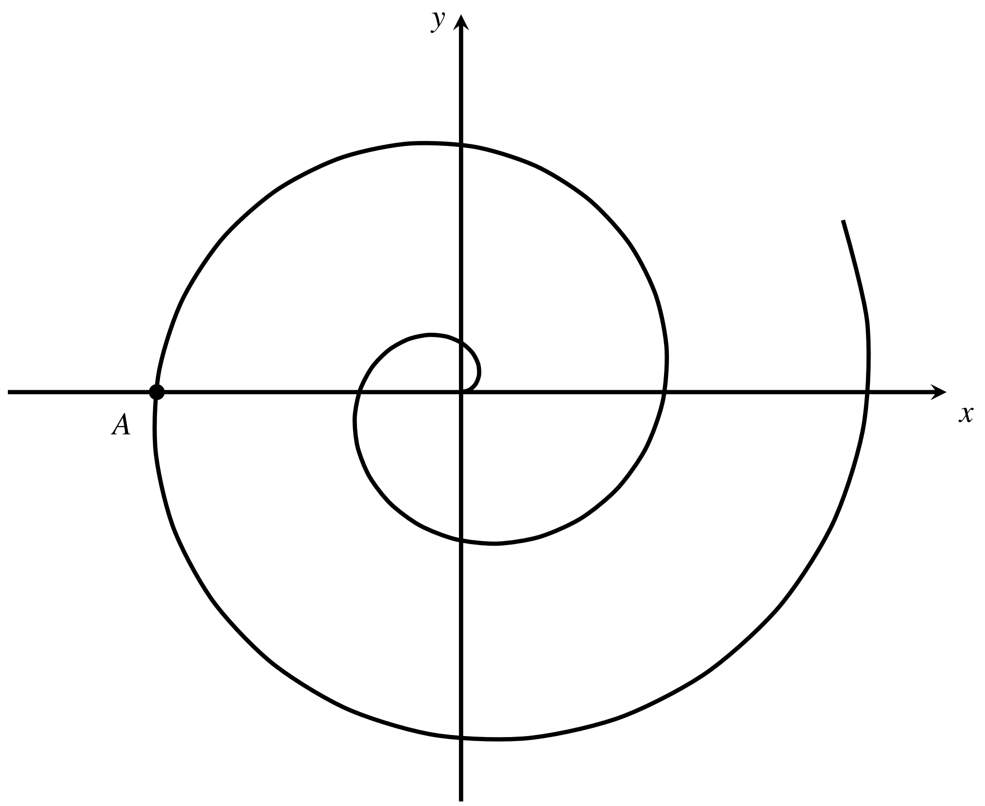
(c) The coefficient of x^2 in the expansion of $(1 - 4x)^n$ is 240, where n is a positive integer.

(i) Show that $n = 6$. **2**

(ii) Hence, find the coefficient of x^4 in the expansion of $(1 - 4x)^n \left(1 + \frac{2}{x}\right)^3$. **3**

(d) The position of a particle at time t is given by the parametric equations $x = t\cos t$, $y = t\sin t$ where $t \geq 0$. **4**

The path of the particle is shown on the diagram below.
By first finding the velocity vector, calculate the exact value of instantaneous speed of the particle at point A.



End of Question 13

Question 14 (15 marks) Use a SEPARATE Writing Booklet

(a) In a square with sides of unit length, 51 points are chosen. **3**

Show that there are at least 3 points which lie within a circle with a radius of $\frac{1}{5\sqrt{2}}$.

(b) Use the principle of mathematical induction to show that $n^3 + 5n$ is divisible by 6 for all positive integers n. **3**

(c) The polynomial $P(x)$ is of degree 3 and $P(x) - 1$ is divisible by $(x - 1)^2$.

(i) Find the value of $P(1)$. **1**

(ii) Show that $P'(x)$ is divisible by $(x - 1)$. **1**

(iii) Given also that $P'(x)$ is divisible by $(x + 1)$ and $P(-1) = -1$, find $P(x)$. **3**

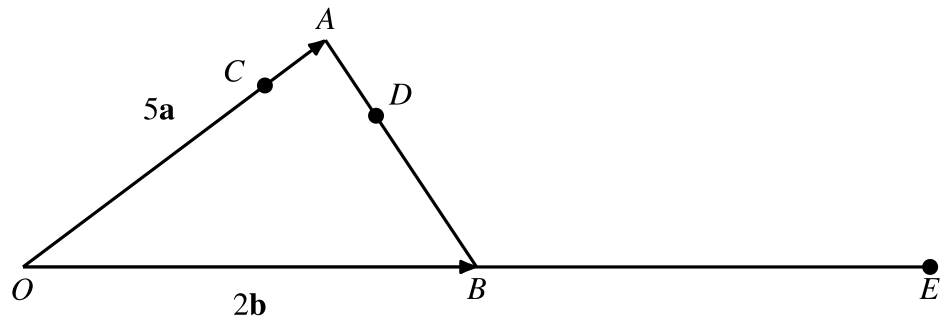
Question 14 continues on page 15

(d) On the diagram below it is given that $\overrightarrow{OA} = 5\mathbf{a}$ and $\overrightarrow{OB} = 2\mathbf{b}$. **4**

C is the point on OA such that $OC : CA = 4 : 1$.

D is the point on AB such that $AD : DB = 1 : 2$.

The line OB is extended to point E.



Given that C, D and E are collinear show that $\overrightarrow{BE} = 2\mathbf{b}$.

End of paper

Multiple choice

Wednesday, 24 July 2024 11:13 AM

1) $|2-3x| \leq 4$

$-4 \leq 2-3x \leq 4$

$-6 \leq -3x \leq 2$

$2 \geq x \geq -\frac{2}{3}$

$\therefore$ (A)

2) $x = 2\cos t$ $y = 2\cos 2t$

$= 2(2\cos^2 t - 1)$

$= 4\cos^2 t - 2$

$= (2\cos t)^2 - 2$

$= x^2 - 2$

(B)

3) $\int \sin^2 3x \, dx$ $\cos 6x = 1 - 2\sin^2 3x$

$= \frac{1}{2}\int 1 - \cos 6x \, dx$ $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$

$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right] + c$

$\therefore$ (D)

4) $\cos^{-1}(-p) + \sin^{-1}(-p)$

$= \pi - \cos^{-1} p - \sin^{-1} p$

$= \pi - (\cos^{-1} p + \sin^{-1} p)$

$= \pi - \frac{\pi}{2}$

$= \frac{\pi}{2}$

$\therefore$ (C)

5) $f(x) = x^2 - 4x + 2$

for inverse:

$x = y^2 - 4y + 2$

$x + 2 = y^2 - 4y + 4$

$x + 2 = (y-2)^2$

$y - 2 = \pm\sqrt{x+2}$

$y = 2 \pm \sqrt{x+2}$

$= 2 + \sqrt{x+2}$

$D_{f(x)}: x \geq 2$

$\therefore R_{f^{-1}(x)}: y \geq 2$

$\therefore$ (B)

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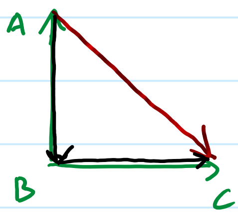
6) (C)

7) AB ⊥ BC



$proj_{\vec{AB}} \vec{AC} = \vec{AB}$

∴ (C)

8) (B)

pick any pair from the set of 5 lines
& pick any pair from the set of 7 lines.

9) (D)

10) Place letters of the word PRESS first.
Remaining letters are O, N, I, M, I

_ P _ R _ E _ S _ S _

Place one of the remaining letters - it has 6 choices.
Next letter then has 7 choices etc...

$\therefore$ total # of ways $= \dfrac{6 \times 7 \times 8 \times 9 \times 10}{2!}$ ← for the 2 I's.

$= \dfrac{10!}{5!\,2!}$

(B)

Question 11

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a) $\dfrac{x+1}{x-3} \leq 2 \qquad x \neq 3$

$\dfrac{x+1}{x-3}(x-3)^2 \leq 2(x-3)^2$

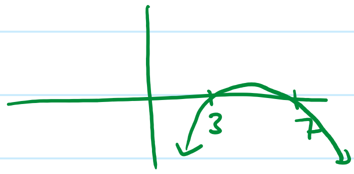
$(x+1)(x-3) \leq 2(x-3)^2$

$(x+1)(x-3) - 2(x-3)^2 \leq 0$

$(x-3)[(x+1) - 2(x-3)] \leq 0$

$(x-3)(x+1-2x+6) \leq 0$

$(x-3)(7-x) \leq 0$



$\therefore \; x < 3, \; x \geq 7$

b) i) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 5 \times 8 - 4 \times 9$

$= 40 - 36$

$= 4$

(ii) $\underset{\sim}{u} \cdot \underset{\sim}{v} > 0 \;\therefore$ the angle between $\underset{\sim}{u}$ & $\underset{\sim}{v}$ is acute.

c) i) From reference sheet

$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

let $A = 2x$, $B = 3x$

$\therefore \cos 2x \cos 3x = \frac{1}{2}[\cos(2x-3x) + \cos(2x+3x)]$

$2\cos 2x \cos 3x = \cos(-x) + \cos 5x$

$= \cos x + \cos 5x$ as $\cos x$ is even.

(ii) $\cos x + \cos 3x = 0$

$2\cos 2x \cos 3x = 0$ (using part (i))

$\cos 2x \cos 3x = 0$

$\cos 2x = 0$, $\cos 3x = 0 \qquad 0 \leq x \leq \pi$

$2x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad 3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4} \qquad x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

d) i) $\alpha+\beta+\alpha+\beta = -\frac{-8}{1}$

$2(\alpha+\beta) = 8$

$\alpha+\beta = 4$

$\alpha = 4-\beta$ — ①

$\alpha\beta(\alpha+\beta) = -\frac{-16}{1}$

$\alpha\beta(\alpha+\beta) = 16$

$\alpha\beta \times 4 = 16$

$\alpha\beta = 4$

sub in ①

$(4-\beta)\beta = 4$

$-\beta^2+4\beta = 4$

$\beta^2-4\beta+4 = 0$

$(\beta-2)^2 = 0$

$\beta = 2 \quad \therefore \alpha = 4-2 = 2$

ii) $\alpha\beta+\alpha(\alpha+\beta)+\beta(\alpha+\beta) = \frac{q}{1}$

$2\times2+2(2+2)+2(2+2) = q$

$q = 20$

e) i) $(-\underset{\sim}{i}-5\underset{\sim}{j})-(7\underset{\sim}{i}-\underset{\sim}{j}) = -8\underset{\sim}{i}-4\underset{\sim}{j}$

$\therefore \underset{\sim}{v} = \frac{1}{2}(-8\underset{\sim}{i}-4\underset{\sim}{j})$

$= -4\underset{\sim}{i}-2\underset{\sim}{j}$

ii) $\underset{\sim}{r} = 7\underset{\sim}{i}-\underset{\sim}{j}+(-4\underset{\sim}{i}-2\underset{\sim}{j})t$

$= 7\underset{\sim}{i}-\underset{\sim}{j}-4t\underset{\sim}{i}-2t\underset{\sim}{j}$

$= (7-4t)\underset{\sim}{i}-(1+2t)\underset{\sim}{j}$

iii) distance to coastguard $= \sqrt{(7-4t)^2+[-(1+2t)]^2}$

$= \sqrt{49-56t+16t^2+1+4t+4t^2}$

$= \sqrt{20t^2-52t+50}$

min distance occurs when $20t^2-52t+50$ is minimum, this occurs at the vertex as it is a concave up parabola.

$t = -\frac{-52}{2\times20}$

$= \frac{52}{40}$

$=$ 1 hour 18 mins

$\therefore$ boat is closest to coastguard at 11:18 am.

OR

Shortest distance is perp distance

$\begin{pmatrix} 7-4t \\ -1-2t \end{pmatrix}\cdot\begin{pmatrix} -4 \\ -2 \end{pmatrix} = 0$

$-4(7-4t)-2(-1-2t) = 0$

$-28+16t+2+4t = 0$

$20t = 26$

$t = \frac{26}{20}$

Question 12

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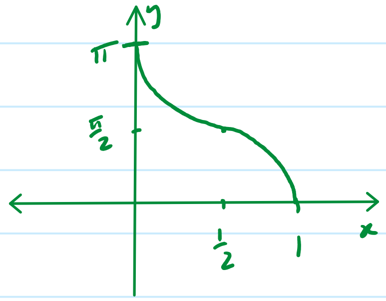
a)



$-1 \le 2x-1 \le 1$

$0 \le 2x \le 2$

$0 \le x \le 1$

b) $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

$= \frac{1}{2}\int \frac{2}{\sqrt{9-(2x)^2}}\,dx$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$

c) $LHS = \frac{\tan a}{\tan 2a - \tan a}$

$= \dfrac{\tan a}{\frac{2\tan a}{1-\tan^2 a} - \tan a}$ $\times (1-\tan^2 a)$ / $\times (1-\tan^2 a)$

$= \frac{\tan a(1-\tan^2 a)}{2\tan a - \tan a(1-\tan^2 a)}$

$= \frac{\tan a(1-\tan^2 a)}{2\tan a - \tan a + \tan^3 a}$

$= \frac{\tan a(1-\tan^2 a)}{\tan a + \tan^3 a}$

$= \frac{\tan a(1-\tan^2 a)}{\tan a(1+\tan^2 a)}$

$= \frac{1-\tan^2 a}{1+\tan^2 a}$

$= \cos 2a$

From reference sheet

if $t = \tan\frac{A}{2}$ $\cos A = \frac{1-t^2}{1+t^2}$

let $A = 2a$

$\therefore$ $t = \tan a$ $\therefore \cos 2a = \frac{1-t^2}{1+t^2}$

$= \frac{1-\tan^2 a}{1+\tan^2 a}$

d) i) $\sqrt{3}\sin 3t - \cos 3t \equiv R\sin 3t\cos\alpha - R\cos 3t\sin\alpha$

Equating coefficients

$R\cos\alpha = \sqrt{3}$ — ①

$R\sin\alpha = 1$ — ②

②÷① $\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

①^2+②^2 $(R\cos\alpha)^2 + (R\sin\alpha)^2 = (\sqrt{3})^2 + 1^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 4$

$R^2(\sin^2\alpha + \cos^2\alpha) = 4$

$R^2 = 4$

$R = 2$ $(R>0)$

$\therefore \sqrt{3}\sin 3t - \cos 3t = 2\sin\left(3t - \frac{\pi}{6}\right)$

ii) $\sqrt{3}\sin 3t - \cos 3t = 0$ $\qquad 0 \le t \le \pi$

$2\sin\left(3t - \frac{\pi}{6}\right) = 0$ $\qquad 0 \le 3t \le 3\pi$

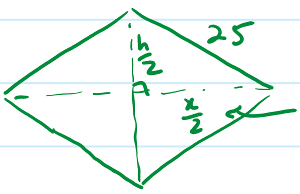
$\sin\left(3t - \frac{\pi}{6}\right) = 0$ $\qquad -\frac{\pi}{6} \le 3t - \frac{\pi}{6} \le \frac{17\pi}{6}$

$3t - \frac{\pi}{6} = 0, \pi, 2\pi,$

$3t = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$

$t = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$

e) i)



diagonals of a rhombus bisect at right angles.

$\left(\frac{h}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = 25^2$ (Pythagoras)

$\frac{h^2}{4} + \frac{x^2}{4} = 625$

$h^2 + x^2 = 4 \times 625$

$h^2 = 2500 - x^2$

$h = \sqrt{2500 - x^2}$ $(h > 0)$

ii) $\frac{dx}{dt} = -0.3$

$h = (2500 - x^2)^{\frac{1}{2}}$

$\frac{dh}{dx} = \frac{1}{2}(2500 - x^2)^{-\frac{1}{2}} \times (-2x)$

$= -x(2500 - x^2)^{-\frac{1}{2}}$

$\frac{dh}{dt} = \frac{dh}{dx} \times \frac{dx}{dt}$

$= \frac{-x}{\sqrt{2500 - x^2}} \times -0.3$

when $x = 30$

$\frac{dh}{dt} = \frac{-30}{\sqrt{2500 - 30^2}} \times -0.3$

$= \frac{9}{40}$ cm/s

Question 13

a) $x^2+(y-10)^2=37 \quad \therefore x^2=37-(y-10)^2$

$$V=\pi\times1^2\times4+\pi\int_4^9 37-(y-10)^2\,dy$$

$$=4\pi+\pi\left[37y-\frac{(y-10)^3}{3}\right]_4^9$$

$$=4\pi+\pi\left[\left(37\times9-\frac{(9-10)^3}{3}\right)-\left(37\times4-\frac{(4-10)^3}{3}\right)\right]$$

$$=4\pi+\pi\left[185-\left(-\tfrac{1}{3}\right)-72\right]$$

$$=\frac{352}{3}\pi \quad u^3$$

b) $$\int_1^e \frac{\sqrt{1+\ln x}}{2x}\,dx$$

let $u=\ln x$

$\frac{du}{dx}=\frac{1}{x}$

$du=\frac{dx}{x}$

$x=1 \Rightarrow u=\ln 1=0$

$x=e \Rightarrow u=\ln e=1$

$$=\int_0^1 \frac{\sqrt{1+u}}{2}\,du$$

$$=\frac{1}{2}\int_0^1 (1+u)^{\frac{1}{2}}\,du$$

$$=\frac{1}{2}\left[\frac{2(1+u)^{\frac{3}{2}}}{3}\right]_0^1$$

$$=\frac{1}{3}\left[(1+u)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{1}{3}\left((1+1)^{\frac{3}{2}}-(1+0)^{\frac{3}{2}}\right)$$

$$=\frac{1}{3}\left(2^{\frac{3}{2}}-1^{\frac{3}{2}}\right)$$

$$=\frac{1}{3}\left(2\sqrt{2}-1\right)$$

c) i) general term: $\binom{n}{r}(-4x)^r\,1^{n-r}=\binom{n}{r}(-4)^r x^r$

when $r=2$, coefficient of $x^2=240$

$\therefore 240=\binom{n}{2}(-4)^2$

$240=\frac{n!}{2!(n-2)!}\times16$

$15=\frac{n(n-1)}{2}$

$30=n(n-1)$

$0=n^2-n-30$

$0=(n-6)(n+5)$

$\therefore n=6,-5$

$\therefore n=6 \quad (n>0)$

ii) general term for 2nd bracket $= \binom{3}{r} 1^{3-r} (2x^{-1})^r$

$= \binom{3}{r} 2^r x^{-r}$

1st bracket	2nd bracket
$\binom{6}{6}(-4)^6 x^6$	$\binom{3}{2} 2^2 x^{-2}$
$\binom{6}{5}(-4)^5 x^5$	$\binom{3}{1} 2^1 x^{-1}$
$\binom{6}{4}(-4)^4 x^4$	$\binom{3}{0} 2^0 x^0$

$\therefore$ coefficient of x^4 is

$= \binom{6}{6}(-4)^6 \binom{3}{2} 2^2 + \binom{6}{5}(-4)^5 \binom{3}{1} 2 + \binom{6}{4}(-4)^4 \binom{3}{0} 2^0$

$= 49152 - 36864 + 3840$

$= 16128$

d) $x = t\cos t$ $\qquad y = t\sin t$

$\dot{x} = \cos t - t\sin t$ $\qquad \dot{y} = \sin t + t\cos t$

$\underset{\sim}{v} = (\cos t - t\sin t)\underset{\sim}{i} + (\sin t + t\cos t)\underset{\sim}{j}$

To find point A set $y = 0$

$0 = t\sin t$

$t = 0$, $\sin t = 0$

$t = 0, \pi, 2\pi, 3\pi \ldots$

A is the 4th time the particle intersects the x-axis

$t = 3\pi$

$x = 3\pi\cos 3\pi$

$= -3\pi$

$\therefore A = (-3\pi, 0)$

$\underset{\sim}{v}(3\pi) = (\cos 3\pi - 3\pi\sin 3\pi)\underset{\sim}{i} + (\sin 3\pi + 3\pi\cos 3\pi)\underset{\sim}{j}$

$= (-1 - 0)\underset{\sim}{i} + (3\pi \times -1)\underset{\sim}{j}$

$= -\underset{\sim}{i} - 3\pi\underset{\sim}{j}$

Speed $= |\underset{\sim}{v}(3\pi)|$

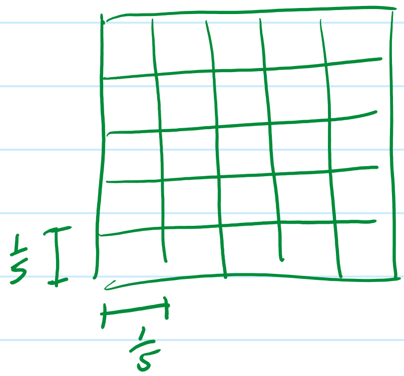
$= \sqrt{(-1)^2 + (-3\pi)^2}$

$= \sqrt{1 + 9\pi^2}$

Question 14

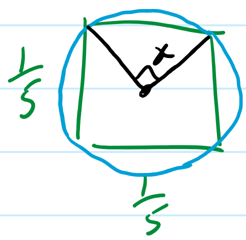
Wednesday, 24 July 2024

a) Take the square of unit length & divide it into a 5 × 5 grid to generate 25 squares.



There are 51 points so by the pigeonhole principle at least 3 points must be in the same square. $51 \div 25 = 2 \text{ r } 1$

Consider one of the squares & inscribe it in a circle.



Thus the radius of the circle is $\frac{1}{2}$ the diagonal of the square.

$$x^2 + x^2 = \left(\frac{1}{5}\right)^2$$
$$2x^2 = \frac{1}{25}$$
$$x^2 = \frac{1}{50}$$
$$x = \frac{1}{\sqrt{50}}$$
$$= \frac{1}{5\sqrt{2}}$$

∴ as the square is inscribed in the circle, at least 3 points must lie inside a circle with radius $\frac{1}{5\sqrt{2}}$.

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b) Prove true for $n=1$:

$1^3+5\times 1$

$=1+5$

$=6$

which is divisible by 6 $\therefore$ true for $n=1$

Assume true for $n=k$ $k\in\mathbb{Z}$

$k^3+5k=6M$, $M\in\mathbb{Z}$

Prove true for $n=k+1$

RTP: $(k+1)^3+5(k+1)=6N$ $N\in\mathbb{Z}$

$LHS=(k+1)^3+5(k+1)$

$=k^3+3k+3k^2+1+5k+5$

$=(k^3+5k)+3k+3k^2+6$

$=6M+3k+3k^2+6$ (using assumption)

$=6(M+1)+3k(k+1)$

Since $M\in\mathbb{Z}$ $6(M+1)$ is divisible by 6.

$3k(k+1)$ is divisible by 3 as 3 is a factor.

k & $k+1$ are consecutive numbers so at least one of them must be even

$\therefore$ $3k(k+1)$ is divisible by both 3 & 2 i.e divisible by 6

$\therefore$ $LHS=6(M+1)+3k(k+1)$

$=6(M+1)+6P$ $P\in\mathbb{Z}$ & $2P=k(k+1)$

$=6(M+1+P)$

$=6N$ $N=M+1+P\in\mathbb{Z}$

$\therefore$ statement is true for all positive integers n by the principle of mathematical induction.

c) i) $P(x) - 1 = (x-1)^2 Q(x)$ where $Q(x) \neq 0$

$P(x) = (x-1)^2 Q(x) + 1$

$P(1) = (1-1)^2 Q(1) + 1$

$= 1$

ii) $P(x) = (x-1)^2 Q(x) + 1$

$P'(x) = 2(x-1) Q(x) + (x-1)^2 Q'(x)$

$= (x-1)\left[2Q(x) + (x-1) Q'(x)\right]$

$\therefore$ $P'(x)$ is also divisible by $(x-1)$

iii) $P'(x)$ is divisible by $(x-1)$ & $(x+1)$

$\therefore P'(x) = k(x-1)(x+1)$

$= k(x^2-1)$

$P(x) = k\left(\frac{x^3}{3} - x\right) + c$

$P(1) = k\left(\frac{1}{3} - 1\right) + c$ $\quad$ $P(-1) = k\left(\frac{-1}{3} - (-1)\right) + c$

$1 = k\left(-\frac{2}{3}\right) + c$ — ① $\quad$ $-1 = k\left(\frac{2}{3}\right) + c$ — ②

① + ② $\quad 0 = 2c$

$c = 0$

$1 = k\left(-\frac{2}{3}\right) + 0$

$1 = -\frac{2}{3}k$

$k = -\frac{3}{2}$

$\therefore P(x) = -\frac{3}{2}\left(\frac{x^3}{3} - x\right)$

d) $\overrightarrow{AB} = 2\underline{b} - 5\underline{a}$

$\therefore \overrightarrow{AD} = \frac{1}{3}(2\underline{b} - 5\underline{a})$

$\overrightarrow{CA} = \underline{a}$

$\overrightarrow{CD} = \overrightarrow{CA} + \overrightarrow{AD}$

$= \underline{a} + \frac{1}{3}(2\underline{b} - 5\underline{a})$

$= \underline{a} + \frac{2}{3}\underline{b} - \frac{5}{3}\underline{a}$

$= -\frac{2}{3}\underline{a} + \frac{2}{3}\underline{b}$

$= -\frac{2}{3}(\underline{a} - \underline{b})$

CDE are collinear $\therefore$ $\overrightarrow{CE}$ is a scalar multiple of $-\frac{2}{3}(\underline{a} - \underline{b})$

i.e it is a scalar multiple of $\underline{a} - \underline{b}$

$\overrightarrow{CE} = \overrightarrow{CO} + \overrightarrow{OE}$

$= -4\underline{a} + (2\underline{b} + \overrightarrow{BE})$

$= -4\underline{a} + (2\underline{b} + k\underline{b})$ where $\overrightarrow{BE} = k\underline{b}$ $(k \in \mathbb{R})$ as OBE are collinear.

$= -4\underline{a} + (2+k)\underline{b}$

since $\overrightarrow{CE}$ is a scalar multiple of $\underline{a} - \underline{b}$

$\overrightarrow{CE} = \lambda(\underline{a} - \underline{b})$

$\therefore \lambda(\underline{a} - \underline{b}) = -4\underline{a} + (2+k)\underline{b}$

$\lambda\underline{a} - \lambda\underline{b} = -4\underline{a} + (2+k)\underline{b}$

equating coefficients of $\underline{a}$ & $\underline{b}$ as they are non-zero & non-parallel

$\lambda = -4$, $\lambda = -(2+k)$

$-4 = -2 - k$

$k = -2 + 4$

$k = 2$

$\therefore \overrightarrow{BE} = 2\underline{b}$